GATE 2027 EC, Section 2B

Signals and systems

LTI systems, Fourier series and transform, sampling, DTFT, DFT and FFT, z-transform and digital filter design.

42 formulas in 8 topics. Equations are numbered section.topic.formula, for example (2B.1.1).

2B.1 Signals and LTI systems

Energy and power (2B.1.1)

$$E = \int |x(t)|^2\,dt;\quad P = \lim (1/2T)\int_{-T}^{T} |x(t)|^2\,dt$$

Even / odd parts (2B.1.2)

$$x_e(t) = [x(t) + x(-t)]/2;\quad x_o(t) = [x(t) - x(-t)]/2$$

Convolution (2B.1.3)

$$y(t) = \int x(\tau)h(t-\tau)d\tau;\quad y[n] = \Sigma\, x[k]h[n-k]$$

Length of discrete convolution (2B.1.4)

$$N_1 + N_2 - 1$$

Causal (2B.1.5)

$$h(t) = 0 \text{ for } t < 0$$

BIBO stable (2B.1.6)

$$\int |h(t)|dt < \infty;\quad \Sigma|h[n]| < \infty$$

Group and phase delay (2B.1.7)

$$\tau_g = -d\varphi/d\omega;\quad \tau_p = -\varphi(\omega)/\omega$$

Distortionless system (2B.1.8)

$|H(\omega)|$ constant and $\varphi(\omega) = -\omega t_0$ (linear phase)

2B.2 Fourier series

Exponential form (2B.2.1)

$$x(t) = \Sigma\, c_k\, e^{jk\omega_0 t};\quad c_k = (1/T)\int_T x(t)e^{-jk\omega_0 t}dt$$

Parseval (2B.2.2)

$$(1/T)\int_T |x(t)|^2 dt = \Sigma|c_k|^2$$

Symmetry (2B.2.3)

Even → cosine terms only; odd → sine terms only; half-wave symmetric → odd harmonics only

Square wave (±A, 50%) (2B.2.4)

$$x(t) = (4A/\pi)\,\Sigma_{k\ \text{odd}}\, \sin(k\omega_0 t)/k$$

2B.3 Fourier transform

Definition (2B.3.1)

$$X(\omega) = \int x(t)e^{-j\omega t}dt;\quad x(t) = (1/2\pi)\int X(\omega)e^{j\omega t}d\omega$$

Pairs (2B.3.2)

$$\delta(t) \leftrightarrow 1;\ 1 \leftrightarrow 2\pi\delta(\omega);\ e^{-at}u(t) \leftrightarrow 1/(a + j\omega);\ e^{-a|t|} \leftrightarrow 2a/(a^2 + \omega^2)$$

Rectangular pulse (2B.3.3)

Width T, height 1 $\leftrightarrow$ $T\cdot\sin(\omega T/2)/(\omega T/2)$

Tones (2B.3.4)

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0);\ \cos\omega_0 t \leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Properties (2B.3.5)

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0}X;\ x(at) \leftrightarrow X(\omega/a)/|a|;\ dx/dt \leftrightarrow j\omega X;\ x\cdot e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Convolution / multiplication (2B.3.6)

$$x * h \leftrightarrow X\cdot H;\quad x\cdot y \leftrightarrow (1/2\pi)\ X * Y$$

Parseval (2B.3.7)

$$\int |x(t)|^2 dt = (1/2\pi)\int |X(\omega)|^2 d\omega$$

2B.4 Sampling

Nyquist (2B.4.1)

$$f_s \geq 2f_{max}\ (\text{Nyquist rate} = 2f_{max})$$

Sampled spectrum (2B.4.2)

$$X_s(f) = f_s\,\Sigma\, X(f - kf_s)$$

Reconstruction (2B.4.3)

Ideal low-pass filter, cut-off between f_{max} and $f_s - f_{max}$, gain T_s

Aliased frequency (2B.4.4)

A tone f appears at $|f - kf_s|$ for the k that brings it into $[0, f_s/2]$

2B.5 Laplace and system functions

Transfer function (2B.5.1)

$H(s) = Y(s)/X(s)$ with zero initial conditions

Stability (causal) (2B.5.2)

All poles in left half-plane; ROC is right of the rightmost pole

Frequency response (2B.5.3)

$H(j\omega) = H(s)$ at $s = j\omega$ (if ROC includes $j\omega$ axis)

2B.6 Discrete-time: DTFT, DFT, FFT

DTFT (2B.6.1)

$X(e^{j\omega}) = \Sigma\, x[n]e^{-j\omega n}$; periodic with period 2π

DFT (2B.6.2)

$X[k] = \Sigma_{n=0}^{N-1}\, x[n]e^{-j2\pi kn/N}$; $x[n] = (1/N)\Sigma\, X[k]e^{j2\pi kn/N}$

Circular convolution (2B.6.3)

$x_1 \circledast x_2 \leftrightarrow X_1[k]X_2[k]$; zero-pad to $\geq N_1 + N_2 - 1$ for linear convolution

Frequency resolution (2B.6.4)

$\Delta f = f_s/N$

FFT cost (radix-2) (2B.6.5)

$(N/2)\log_2 N$ complex multiplications vs N^2 for direct DFT

DFT symmetry (real x) (2B.6.6)

$X[N-k] = X^*[k]$

2B.7 z-transform

Definition (2B.7.1)

$X(z) = \Sigma\, x[n]z^{-n}$

Key pairs (2B.7.2)

$\delta[n] \leftrightarrow 1$; $a^n u[n] \leftrightarrow 1/(1-az^{-1})$, $|z| > |a|$; $-a^n u[-n-1] \leftrightarrow 1/(1-az^{-1})$, $|z| < |a|$

Properties (2B.7.3)

$x[n-k] \leftrightarrow z^{-k}X(z)$; $n \cdot x[n] \leftrightarrow -z\, dX/dz$; $a^n x[n] \leftrightarrow X(z/a)$

Stability (causal) (2B.7.4)

All poles inside the unit circle; ROC includes $|z| = 1$

Final value (2B.7.5)

$x[\infty] = \lim_{z\to 1}(z-1)X(z)$

2B.8 Digital filters

Linear-phase FIR (2B.8.1)

$h[n] = \pm h[N-1-n]$ (symmetric or antisymmetric)

Bilinear transform (2B.8.2)

$s = (2/T)(1-z^{-1})/(1+z^{-1})$; prewarp $\Omega = (2/T)\tan(\omega/2)$

Impulse invariance (2B.8.3)

$\omega = \Omega T$; suffers aliasing, use for low-pass/band-pass only

Butterworth (2B.8.4)

$|H(j\Omega)|^2 = 1/(1+(\Omega/\Omega_c)^{2N})$; -3 dB at Ω_c; roll-off $20N$ dB/decade

Window main-lobe width (2B.8.5)

Rectangular $4\pi/N$; Hanning $8\pi/N$; Hamming $8\pi/N$; Blackman $12\pi/N$

My notes